# Math 112: Introductory Real Analysis

Last time: limits of functions $\lim_{x \to p} f(x) = q$

continuity $\begin{cases} \lim_{x \to p} f(x) = f(p) \\ f^{-1}(U) \text{ is open, for all open } U \end{cases}$

Today: Continuity and compactness

<u>Thm</u> Suppose $f: X \to Y$ is a continuous map between metric spaces $X$ and $Y$.
Then for any compact $K \subseteq X$, $f(K) \subseteq Y$ is compact.

proof) Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(K)$.
Then, since $f$ is continuous, $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of $K$.
Since $K$ is compact, there is a finite subset $J \subseteq I$ (with $J = \{\alpha_1, \dots, \alpha_n\}$) such that

$$K \subseteq f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n}).$$

It follows that $f(K) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$, i.e. we have found a finite subcover.
Therefore, $f(K)$ is compact. ∎

**Cor** If $X$ is a compact metric space and $f: X \to \mathbb{R}^k$ is a continuous map,

then $f(X)$ is closed and bounded.

(Extreme Value Theorem)

**Cor** Suppose $f$ is a continuous real function on a compact metric space $X$.

Then $f$ attains its minimum and maximum.

(In other words, if we set $M := \sup_{p \in X} f(p)$ and $m := \inf_{p \in X} f(p)$,

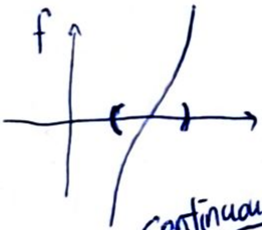
then there exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.)

proof) $f(X) \subset \mathbb{R}$ is closed and bounded,

so $f(X)$ contains $M := \sup f(X)$ and $m := \inf f(X)$. ■

**Rmk** A continuous, real function $f$ on a non-compact metric space $X$ may not attain maximum or minimum.

e.g.




**Thm** Suppose $f$ is a continuous bijective map from a compact metric space $X$ to a metric space $Y$.

Then, the inverse map $f^{-1}: Y \to X$ is continuous.

proof) To show that $f^{-1}: Y \to X$ is continuous, it suffices to show that

for every closed $V \subseteq X$, $(f^{-1})^{-1}(V) = f(V) \subseteq Y$ is closed.

This is true because $V$ is compact (being a closed subset of compact $X$),

and hence $f(V)$ is compact (and in particular closed). ■

**Rmk** In the previous theorem, compactness of $X$ is necessary.

For instance, $f : [0,1) \longrightarrow S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$

$$\theta \longmapsto e^{2\pi i \theta}$$

is continuous and bijective, but $f^{-1}$ is not continuous.

**Def** Let $f : X \to Y$ be a map between metric spaces.
We say that $f$ is *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \varepsilon$ for all $p, q \in X$ for which $d_X(p,q) < \delta$.

* Note that $\delta$ is chosen "globally" or "uniformly" over $X$.
Uniform continuity is stronger than continuity of $f$, which is a local condition.

**Thm** Let $f : X \to Y$ be a continuous map from a compact metric space $X$ to a metric space $X$. Then $f$ is uniformly continuous.

proof) Let $\varepsilon > 0$ be given. Since $f$ is continuous, for each point $p \in X$, there is a positive number $\delta_p$ such that

$$q \in X,\ d_X(p,q) < \delta_p \text{ implies } d_Y(f(p), f(q)) < \frac{\varepsilon}{2}.$$

Now, consider the open cover $\{ B_{\frac{\delta_p}{2}}(p) \}_{p \in X}$ of $X$. Since $X$ is compact, it must have a finite subcover $\{ B_{\frac{\delta_{p_i}}{2}}(p_i) \}_{i \in \{1, \cdots, n\}}$.

Put $\delta := \frac{1}{2} \min \{ \delta_{p_1}, \cdots, \delta_{p_n} \} > 0$.

 (proof continued)

Now if $p, q \in X$ are points such that $d_X(p,q) < \delta$,

then $p \in B_{\frac{\delta_{p_i}}{2}}(p_i)$ for some $p_i \in \{p_1, \cdots, p_n\}$, and hence

$$d_X(p, p_i) < \frac{\delta_{p_i}}{2} < \delta_{p_i}.$$

We also have

$$d_X(q, p_i) \leqslant d_X(p,q) + d_X(p, p_i) < \delta + \frac{\delta_{p_i}}{2} < \delta_{p_i}.$$

Therefore,

$$d_Y(f(p), f(q)) \leqslant d_Y(f(p), f(p_i)) + d_Y(f(q), f(p_i)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. ■

Ex On non-compact spaces, there are continuous but not uniformly continuous functions.

e.g. $f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^2$ is continuous but not uniformly continuous.